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A tensor theory of gravitation in a curved metric on a flat background

J E Drummond†

Department of Mathematics, University of Aston in Birmingham, Birmingham, B4 7ET, England

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Abstract. A theory of gravity is proposed using a tensor potential for the field on a flat metric. This potential cannot be isolated by local observations, but some details can be deduced from measurements at a distance. The requirement that the field equations for the tensor potential shall be deducible from an action integral, that the action and field equations are gauge invariant, and, conversely, that the Lagrangian in the action integral can be integrated from the field equations leads to Einstein's field equations. The requirement that the field energy-momentum tensor exists leads to a constraint on the tensor potential. If the constraint is a differential gauge condition, then it can only be the Hilbert condition giving a unique background tensor, metric tensor and tensor potential. For a continuous field inside a solid sphere the metric must be homogeneous in the spatial coordinates, and the associated field energy-momentum tensor has properties consistent with Newtonian dynamics.

1. Introduction

R H Dicke on page 211 of DeWitt and DeWitt's (1964) conference report comments that it is remarkable that gravity is interpreted as a manifestation of Riemannian geometry whereas all other forces are treated as the effects of particle interactions. This is not a serious criticism, since mechanics may be formulated in terms of Newtonian force laws, Lagrangian dynamics, Hamiltonians, action integrals or geodesics, and all of these formulations are equivalent. In fact Misner *et al* (1973) give on pages 417–28 six different ways of deriving Einstein's equations.

A more serious objection to Einstein's theory of gravitation is that it is incomplete and deals with parameters or generalised coordinates. Thus both the Schwarzchild and homogeneous solutions of the field equations are equally correct and differ only in the radial parameter. The two- and three-body problems are also difficult to comprehend in generalised coordinates.

The desire for at least an inertial frame of reference, if not an absolute frame or a flat earth, is expressed in the linear Lorentz invariant tensor theories of gravitation by Birkhoff (1943, 1944) supported by Barajas (1944) and Belinfante and Swihart (1957). However, Weyl (1944a, b) shows that a linear tensor theory on flat space–time must be a first-order approximation to Einstein's theory, and the linear theory is inconsistent in the second-order terms.

† Present address: Australian National University, Canberra, Australia.

Gupta (1957) showed that Einstein's equations could be expanded out as a power series on a flat metric, and Deser (1970) sketched a proof that, starting from a flat metric, self-consistency led to Einstein's equations and the non-observability of the background metric.

Roxborough and Tavakol (1978) showed that the particle and field equations of general relativity may be expressed in any geometry if the residual terms are interpreted as arising from a force field.

Thirring (1961) gives a non-linear theory of scalar, vector and tensor fields on a flat background working with both flat and renormalised space, starting with a field theory Lagrangian and a gauge condition, then appealing to Riemannian geometry to choose nonlinear terms. Cavalleri and Spinelli (1974a, b, 1975) develop a similar tensor field theory, but staying in flat space and developing other aspects of the theory. They finally use Deser's (1970) method to conclude that their theory is equivalent to Einstein's. Petry (1975), on the other hand, gives a linear tensor field theory in flat space with a particle equation containing an arbitrary quadratic term not derivable from the field equations, so this must be rejected on the grounds of inconsistency.

The following is a tensor field theory similar to those of Thirring (1961) and Cavalleri and Spinelli (1974a, b, 1975), but working in the curved metric and yet giving reason for believing that there is a preferred background metric. This background metric is not locally observable as was concluded by Deser (1970) and Roxborough and Tavakol (1978), but can be deduced from measurements at a distance as described in appendix 2 assuming there is a flat outer metric and that a field energy–momentum exists.

2. The equations of motion of a particle

Of the different forms of mechanics the Lagrangian is fairly central, so let us start with a Lagrangian in four-space,

$$L=\frac{1}{2}mc^2 b_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\mathrm{d}x^j}{\mathrm{d}s}+\frac{1}{2}mc^2 h_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\mathrm{d}x^j}{\mathrm{d}s}+qA_i\frac{\mathrm{d}x^i}{\mathrm{d}s}+zV, \tag{1}$$

where m is the mass of the particle, q is the electric charge on the particle, and z is the measure of some other property associated with the scalar field potential V. A_i is the vector electromagnetic field potential, and h_{ij} is a tensor potential which I wish to identify as the gravitational field potential. Also c is the fundamental Lorentzian velocity (velocity of light in free space), and b_{ij} is the metric tensor of a flat background metric such as the Minkowskian diag $(c^2, -1, -1, -1)$ or a functionally related set of coordinates such as spherical polar coordingtes. s is a timelike parameter to be defined in § 2.1.

In this discussion I wish to concentrate on the tensor potential and do not wish to assign any properties to V other than to suggest that it could be the potential of a short-range scalar nuclear force such as the Yukawa potential. We could also add to the Lagrangian other terms related to the spin and other nuclear properties.

The properties of the scalar potential have been dealt with by Thirring (1961). The properties of the vector potential are covered in a number of textbooks on electromagnetism and special relativity and, if we convert from ordinary to covariant derivatives, the same formulae may be used in general relativity.

In order to write down the Lagrange equations let us assume that b_{ij}, h_{ij}, A_i and V are functions of the four-coordinates, x^i, while m, c, q and z are constants. Let $g_{ij} = b_{ij} + h_{ij}$, then the Euler or Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{2}mc^2\left(g_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}+g_{ki}\frac{\mathrm{d}x^k}{\mathrm{d}s}\right)+qA_i\right]=\frac{\partial}{\partial x^i}\left(\frac{1}{2}mc^2 g_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s}+qA_j\frac{\mathrm{d}x^j}{\mathrm{d}s}+zV\right). \qquad (2)$$

These equations may be expanded either as acceleration equations,

$$g_{ij}\frac{\mathrm{d}^2x^j}{\mathrm{d}s^2}=-\frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k}+\frac{\partial g_{ki}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^i}\right)\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s}+\frac{q}{mc^2}\left(\frac{\partial A_j}{\partial x^i}-\frac{\partial A_i}{\partial x^j}\right)\frac{\mathrm{d}x^j}{\mathrm{d}s}+\frac{z}{mc^2}\frac{\partial V}{\partial x^i}, \qquad (3)$$

or momentum equations,

$$\frac{\mathrm{d}p_i}{\mathrm{d}s}=\frac{1}{2}mc^2\frac{\partial g_{jk}}{\partial x^i}\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s}+q\frac{\partial A_j}{\partial x^i}\frac{\mathrm{d}x^j}{\mathrm{d}s}+z\frac{\partial V}{\partial x^i}, \qquad (4)$$

where $p_i = \partial L/\partial(\mathrm{d}x^i/\mathrm{d}s)$.

Furthermore, if the background metric tensor b_{ij} is constant, then $\partial g_{ij}/\partial x^k = \partial h_{ij}/\partial x^k$ and, for a weak field, $\partial h_{ij}/\partial x^k \simeq h_{ij;k}$. The acceleration equations (3) may then be written in vector form by transferring some terms across the equation to give a generalised acceleration

$$g_{ij}\frac{\mathrm{d}^2x^j}{\mathrm{d}s^2}+\left(\Gamma_{jk,i}-G_{jk,i}\right)\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s}$$

which is equal to

$$-G_{jk,i}\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s}+\frac{q}{mc^2}F_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}s}+\frac{z}{mc^2}V_i,$$

where $\Gamma_{jk,i}$ is the Christoffel symbol, $G_{jk,i}=\frac{1}{2}(h_{ij;k}+h_{ik;j}-h_{jk;i})$, $F_{ij}=(A_{j;i}-A_{i;j})$, $V_i = V_{;i}$. The semicolon denotes covariant differentiation using the metric tensor g_{ij}. Similarly the generalised rate of change of momentum is

$$\frac{1}{2}mc^2h_{jk;i}\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s}+qA_{j;i}\frac{\mathrm{d}x^j}{\mathrm{d}s}+zV_{;i} \qquad (5)$$

Here V_i is a vector force field, F_{ij} is the electromagnetic field tensor, and we have a similar gravitation tensor $G_{ij,k}$ which is one stage more complicated than F_{ij}. Thus, while F_{ij} is multiplied by q/mc^2 and a velocity to give an acceleration, $G_{ij,k}$ is multiplied by the product of two velocities to give an acceleration.

On the other hand, there are dissimilarities in that A_i never appears in the equations of motion, and the identity $F_{ij}=A_{j;i}-A_{i;j}$ cannot be solved to find $A_{i;j}$. In contrast to this, h_{ij} is linked with b_{ij} in the coefficient g_{ij}, while the identity

$$G_{ij,k}=\frac{1}{2}(h_{ik;j}+h_{jk;i}-h_{ij;k})$$

can be solved to give us

$$h_{ij;k}=G_{ki,j}+G_{kj,i}.$$

If also h_{ij} can be separated from b_{ij}, as is proposed in §§ 4.1 and 4.2, the equations of motion may be expressed in terms of $G_{ij,k}$, $h_{ij;k}$ and h_{ij}.

2.1. Definition of s

So far s is undefined, but if L does not contain s explicitly we have a first integral which is the Hamiltonian

$$H = \frac{dx^i}{ds}\frac{\partial L}{\partial(dx^i/ds)} - L = \frac{1}{2}mc^2(b_{ij}+h_{ij})\frac{dx^i}{ds}\frac{dx^j}{ds} - zV = \text{constant}. \qquad (6)$$

If we adopt units of length and time, and b_{ij} is the Minkowski metric tensor for some outer region where h_{ij} and V are both zero, and the fundamental velocity in these units is c, we may conveniently standardise s so that $ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2$. Hence the constant in equation (6) is $\frac{1}{2}mc^2$, and

$$ds^2(1+2zV/mc^2) = g_{ij}\,dx^i\,dx^j.$$

It may be noted here that some writers give the mechanical part of the Lagrangian as

$$\left(g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)^{1/2}$$

instead of the quadratic form used in equation (1), possibly because this is associated with the arc length in the action integral. Also the power series expansion of $mc(1-v^2/c^2)^{1/2}$ for small velocities gives the Newtonian kinetic energy as the first non-constant term. However, any action integral of the form

$$\int \left(g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)^n ds$$

leads to the same equation of motion independent of n provided $n \neq 0$. Also in the form of the Lagrangian in equation (1) the amalgamation of $b_{ij}+h_{ij}$ occurs most simply if $n = 1$.

2.2. Non-local observability of the tensor potential

We started with the hypothesis that there was a flat background metric and a tensor potential, and we find that the tensor potential h_{ij} is combined with the background metric b_{ij} to form g_{ij} in both the Lagrangian and in the metric equation. The equation of motion, however, is somewhat different. It is a vector equation and, as shown by Roxborough and Tavakol (1978), it may be arbitrarily separated into two vectors. In making our choice we have carried out a non-local act which requires a knowledge of the metric, assumed flat, in some distant outer region. We further assumed that $h_{ij} = 0$ at a great distance, and that this flat metric could be extended inwards by some geometrical construction or by a knowledge of how to correct our measuring rods and clocks. Hence we can find h_{ij} if we know g_{ij} and b_{ij} and subtract b_{ij} from g_{ij}. Then we can construct the right-hand side of the equation of motion.

If, however, at this stage we make no restriction on coordinates, then the local metric, and hence h_{ij}, is not unique. The proof of this is as follows: Let us make a small displacement of coordinates ξ^i to new coordinates x^{i*}, where $x^{i*} = x^i + \xi^i$. ξ^i is arbitrary apart from having bounded derivatives and by being zero at a large distance from the world tube containing the material of the universe. Under this transformation (see Soper 1976, p 201) the new metric tensor $g_{ij}{}^* = g_{ij} - \xi_{i;j} - \xi_{j;i}$ at the displaced field point x^{i*}. If ξ^i is arbitrary but is zero at a large distance, then b_{ij}, being an extension from outside, is not altered. Hence we may arbitrarily change h_{ij} to $h_{ij} - \xi_{i;j} - \xi_{j;i}$ if no restriction is placed on ξ^i. This is a gauge transformation. Thirring (1961) points out both the similarity and the difference between this and the gauge transformation in electromagnetism, namely that the tensor gauge transformation is linked with the coordinate transformation whereas the vector gauge transformation is not. However, we shall see in § 4 that ξ^i and h_{ij} must be restricted.

3. The general field action

A simple action integral using only the electromagnetic and gravitational terms has the form

$$S = \int (-g)^{1/2}\, \mathrm{d}^4x(\Lambda_2 + \Lambda_1 - c^{-1}A_i\boldsymbol{J}^i - \tfrac{1}{2}g_{ij}\boldsymbol{T}^{ij}), \tag{7}$$

where Λ_2 is a scalar function of first covariant derivatives of h_{ij}, Λ_1 is a scalar function of first covariant derivatives of A_i, J^i is the electric current vector, and T^{ij} is the matter energy–momentum tensor.

Although a background metric tensor b_{ij} is postulated, we saw in § 2.2 that b_{ij} does not appear in the metric equation. Hence, to be consistent in postulating a physical law involving the metric, we must not use b_{ij} but must formulate any law involving the metric using covariant derivatives and using the general metric tensor g_{ij} for all metric properties and for raising and lowering of suffixes on tensors.

Nevertheless, the tensor potential h_{ij} will appear in the force and energy terms together with its covariant derivatives. The use of the general metric causes some complication in reversing second derivatives ($h_{ij;kl}$ may not necessarily be equal to $h_{ij;lk}$ in a general gravitational field), but it avoids some of Thirring's (1961) difficulties in mixing coordinate systems and the complication met by Cavalleri and Spinelli (1974a, b, 1975) in having to postulate a total energy–momentum tensor $T_{ij}^{(\mathrm{tot})}$ to correct for giving the conservation law in the background metric instead of the general metric.

3.1. *The action for a particle*

This may be derived from the total field action for any small body in the field by isolating that part of the action integral which covers the world track of the small body.

For the small body let

$$T^{ij} = \rho c^2 \frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\mathrm{d}x^j}{\mathrm{d}s}, \qquad J^i = \sigma c \frac{\mathrm{d}x^i}{\mathrm{d}s} \qquad \text{and} \qquad (-g)^{1/2}\,\mathrm{d}^4x \to \mathrm{d}\tau_3\,\mathrm{d}s,$$

where ρ is the mass density, σ is the charge density, s is the proper length along the centre of the world track, and $\mathrm{d}\tau_3$ is an element of volume normal to the track. The total mass and charge of the body are then m and q where $m = \int \rho\, \mathrm{d}\tau_3$ and $q = \int \sigma\, \mathrm{d}\tau_3$.

If we set to one side all those terms in the action, including any constant internal stress energy of the small body which do not depend on the path, the action in equation (7) reduces to

$$\int \left(\frac{1}{2}mc^2 g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\mathrm{d}x^j}{\mathrm{d}s} + qA_i\frac{\mathrm{d}x^i}{\mathrm{d}s}\right)\mathrm{d}s$$

as was assumed in equation (1).

3.2. The linear approximation

Following Weyl (1944b) we set down the most general scalar function for Λ_2 which is quadratic in the first derivatives of the tensor potentials,

$$\Lambda_2 = Ah_{ij;k}h^{ij;k} + Bh_{ij;k}h^{ik;j} + Ch^{i}_{i;k}h_j^{j;k} + Dh^{i}_{i;j}h^{jk}{}_{;k} + Eh_i^{j}{}_{;j}h^{ik}{}_{;k}, \tag{8}$$

and similarly for the electromagnetic field,

$$\Lambda_1 = FA_{i;j}A^{i;j} + GA_{i;j}A^{j;i} + HA^{i}_{;i}A^{j}_{;j}.$$

It should be noted here that if we include any second derivatives these may be removed by integration by parts. Thus

$$\int_R h_{ij}h^{ij;l}{}_{l}(-g)^{1/2}\,\mathrm{d}^4x = -\int_R h_{ij;l}h^{ij;l}(-g)^{1/2}\,\mathrm{d}^4x + \int_{\partial R_l} h_{ij}h^{ij;l}\,\mathrm{d}S_l.$$

Hence if the surface integral is taken over some outer boundary, when h_{ij} is zero the term containing a second derivative, namely $h_{ij}h^{ij;l}{}_{l}$ in the action integral, may be replaced simply by $-h_{ij;l}h^{ij;l}$, and the other possible second derivatives may be transformed similarly.

We may also note that in the linear approximation we can, as in appendix 1(iii), interchange the partial derivatives so that the terms $A_{i;j}A^{j;i}$ and $A_i{}^{;i}A^{j}{}_{;j}$ and similarly $h_{ij;k}h^{ik;j}$ and $h_{ij}{}^{;j}h^{ik}{}_{;k}$ make, to second-order accuracy, equal contributions to the action integral.

3.3. The electromagnetic field equations

The electromagnetic equations have been dealt with in a number of textbooks on electricity and special relativity and books on field theory, so it suffices to list a few of their properties for comparison with the gravitational equations:

The Langrangian Λ_1 where

$$\Lambda_1 = \frac{1}{16\pi}F_{ij}F^{ij} = \frac{1}{8\pi}(A_{i;j}A^{i;j} - A_{i;j}A^{j;i})$$

is gauge invariant.

The field equation $(F^{ij})_{;j} = (4\pi/c)J^i$ is gauge invariant.

The field equation is derivable from the action integral and, conversely, if we multiply $(F^{ji})_{;j}$ by the vector potential A_i and integrate by parts, we recover the action integral!

A field energy–momentum tensor exists.

The electromagnetic field equations are completely expressible in terms of the field tensor F_{ij} without the explicit use of the vector potential or its first derivative.

4. The linearised gravitational field equations

To obtain a field equation which is linear in h_{ij} and contains second derivatives with respect to the coordinates we start with an action integral which is quadratic in the first derivatives of the h_{ij}. While recognising that this is only basically linear because further terms in h_{ij} are also concealed in the g_{ij}, we start, omitting the electric terms from equation (7), with an action integral

$$S=\int(\Lambda_2-\tfrac{1}{2}g_{ij}T^{ij})(-g)^{1/2}\,\mathrm{d}^4x, \tag{9}$$

a constant total energy constraint

$$M=\int T(-g)^{1/2}\,\mathrm{d}^4x,$$

and a symmetry constraint $h_{ij}=h_{ji}$, where $T=T_i^{\ i}$ and Λ_2 is given in equation (8).

Using Lagrange multipliers for the constraints, the Euler equation as given in appendix 1(iv) for an extremal of $S+(2\mu+\frac{1}{2})M$ is

$$-\left(\frac{\partial\Lambda_2}{\partial h_{ij;k}}\right)_{;k}=\frac{1}{2}T^{ij}-\mu g^{ij}T-\frac{1}{2}g^{ij}\Lambda_2-\frac{\partial\Lambda_2}{\partial g_{ij}}$$
$$-\frac{1}{2}\left(h_l^{\ i}\frac{\partial\Lambda_2}{\partial h_{lj;k}}+h_l^{\ j}\frac{\partial\Lambda_2}{\partial h_{li;k}}-h_l^{\ k}\frac{\partial\Lambda_2}{\partial h_{li;j}}-h_l^{\ k}\frac{\partial\Lambda_2}{\partial h_{lj;i}}+h_l^{\ i}\frac{\partial\Lambda_2}{\partial h_{kl;j}}+h_l^{\ j}\frac{\partial\Lambda_2}{\partial h_{kl;i}}\right)_{;k}. \tag{10}$$

If we assume that h_{ij} is small, delete the quadratic terms, and allow the reversing of second derivatives, which according to appendix 1(i) is accurate to first order, then

$$\partial\Lambda_2/\partial h_{ij;k}=2Ah^{ij;k}+B(h^{ik;j}+h^{jk;i})+2Cg^{ij}h_l^{\ l;k}$$
$$+D(g^{ij}h^k{}_l{}^{;l}+\tfrac{1}{2}g^{ik}h_l^{\ l;j}+\tfrac{1}{2}g^{jk}h_l^{\ l;i})+E(g^{jk}h^i{}_l{}^{;l}+g^{ik}h^j{}_l{}^{;l}),$$

and the first-order approximations to the ten field equations are

$$2Ah^{ij;l}{}_l+(B+E)(h^{il;j}{}_l+h^{jl;i}{}_l)+2Cg^{ij}h_l^{\ l}{}_{;m}{}^m+D(g^{ij}h_l^{\ m}{}_{;m}{}^l+h_l^{\ l;ij})+\tfrac{1}{2}T^{ij}-\mu g^{ij}T=0. \tag{11}$$

We may eliminate T from equation (11) by multiplying by g_{ij} and then summing to give

$$(2A+8C+D)h_l^{\ l}{}_{;m}{}^m+(2B+2E+4D)h_l^{\ m}{}_{;m}{}^l+(\tfrac{1}{2}-4\mu)T=0.$$

Hence, eliminating T,

$$2Ah^{ij;l}{}_l+(B+E)(h^{il;j}{}_l+h^{jl;i}{}_l)+2C_1g^{ij}h_l^{\ l}{}_{;m}{}^m+D_1g^{ij}h_l^{\ m}{}_{;m}{}^l+Dh_l^{\ l;ij}+\tfrac{1}{2}T^{ij}=0, \tag{12}$$

where

$$C_1=\frac{C+\mu(2A+D)}{1-8\mu},\qquad D_1=\frac{D+4\mu(B+E)}{1-8\mu}.$$

Equations (9) and (12) must satisfy five criteria:

(i) $T^{ij}{}_{;j}=0$ to at least first order in h_{ij}, hence

$$(2A+B+E)h^{il}{}_{;l}{}^m{}_m+(B+E+D_1)h^{lm;i}{}_{lm}+(2C_1+D)h^l{}_l{}^{;i}{}_m{}^m=0. \tag{13}$$

(ii) The field equation (12) is gauge invariant, and since the gauge is related to the coordinate displacement we change x^i by ξ^i and h_{ij} to $h_{ij}-\xi_{i;j}-\xi_{j;i}$. Hence equation (12) is unaltered if

$$(2A+B+E)(\xi^{i;jl}{}_l+\xi^{j;il}{}_l)+2(B+E+D)\xi^{l;ij}{}_l+(4C_1+2D_1)g^{ij}\xi^l{}_{;ln}{}^n=0. \tag{14}$$

(iii) The action integral (9) is gauge invariant. The simultaneous change of x^i and h_{ij} with constant b_{ij} does not alter T or the volume element $(-g)^{1/2}\,\mathrm{d}^4x$. If we change Λ_2 and integrate by parts to simplify the integral, then the change in S is

$$\int (-g)^{1/2}\,\mathrm{d}^4x\Big[(4A+2B+2E)h_{lm}\xi^{l;mn}{}_n+(2B+2E+2D)h_{lm}\xi^{n;lm}{}_n$$
$$+(4C+2D)h_l{}^l\xi^{m;n}{}_{mn}\Big]=0. \tag{15}$$

To satisfy equations (13), (14) and (15) we require $2A=-B-E=D=D_1=-2C=-2C_1$ and $\mu=0$. Hence the action integral reduces to

$$S+\tfrac{1}{2}M=\int\Big[A(h_{lm;n}h^{lm;n}-2h_{lm;n}h^{ln;m}+2h_l{}^{l;m}h_{mn}{}^{;n}-h_l{}^{l;n}h^m{}_{m;n})$$
$$+E(h_l{}^m{}_{;m}h^{ln}{}_{;n}-h_{lm;n}h^{ln;m})-\tfrac{1}{2}g_{ij}T^{ij}+\tfrac{1}{2}T\Big](-g)^{1/2}\,\mathrm{d}^4x, \tag{16}$$

and the linearised field equations are

$$2A(h^{ij;l}{}_l-h^{il;j}{}_l-h^{jl;i}{}_l+h_l{}^{l;ij}+g^{ij}h_l{}^m{}_{;m}{}^l-g^{ij}h_l{}^l{}_{;m}{}^m)+\tfrac{1}{2}T^{ij}=0. \tag{17}$$

This is the same as the linearised Einstein equations with $A=c^2/64\pi G$.

(iv) Conversely we may recover the Lagrangian Λ_2 by multiplying the potential terms of the field equation (17) by h_{ij} or g_{ij} and integrating by parts with constant b_{ij}; thus

$$2A\int h_{ij}(h^{ij;l}{}_l-h^{il;j}{}_l-h^{jl;i}{}_l+h_l{}^{l;ij}+g^{ij}h_l{}^m{}_{;m}{}^l-g^{ij}h_l{}^l{}_{;m}{}^m)(-g)^{1/2}\,\mathrm{d}^4x \tag{18}$$
$$=-2A\int(h_{lm;n}h^{lm;n}-2h_{lm;n}h^{ln;m}+2h_l{}^{l;m}h_{mn}{}^{;n}-h_l{}^l{}_{;n}h_m{}^{m;n})(-g)^{1/2}\,\mathrm{d}^4x$$
$$=-2\int\Lambda_2(-g)^{1/2}\,\mathrm{d}^4x,$$

provided that the surface integrals are zero. Also if we change the order of derivatives in the second and third terms of the integrand in expression (18) from $h^{il;j}{}_l+h^{jl;i}{}_l$ to $h^{il}{}_{;l}{}^j+h^{jl}{}_{;l}{}^i$, we could recover the term $2h_{lm}{}^{;m}h^l{}_n{}^{;n}$ instead of $2h_{lm;n}h^{ln;m}$. This, however, only becomes important in the second-order terms.

(v) A field energy–momentum tensor exists. To find this tensor we set up two conditions: (a) If t^{ij} is this tensor, then the force per unit volume on the medium is $t^{ij}{}_{;j}$. (b) If the medium carries a material energy–momentum tensor T^{ij}, then the force per unit volume on the medium is, according to equation (5) and § 3.1, $\tfrac{1}{2}h_{jk;i}T^{jk}$; hence

$$\tfrac{1}{2}h_{jk;i}T^{jk}=t^{ij}{}_{;j}. \tag{19}$$

Soper (1976, p 34) and Thirring (1961, p 103) give a tensor which satisfies equation (19) but is not necessarily symmetric in i and j and is not gauge invariant, so Thirring immediately adopts the Hilbert gauge. To find why this may be necessary we substitute for $\tfrac{1}{2}T^{ik}$ from the field equation (12) or (17) and write down the most general symmetric tensor for t^{ij} which is constructed from the products of pairs of derivatives of h_{ij}, namely

$$t^{ij} = g^{ij}(ah_{lm;n}h^{lm;n} + bh_{lm;n}h^{ln;m} + ch_l{}^{l;n}h_m{}^m{}_{;n} + dh_l{}^{l;m}h_{mn}{}^{;n} + eh_l{}^m{}_{;m}h^{ln}{}_{;n})$$
$$+h^{ij}{}_{;l}(fh^l{}_m{}^{;m} + gh_m{}^{m;l}) + (h^i{}_l{}^{;j} + h^j{}_l{}^{;i})(hh^l{}_m{}^{;m} + ih_m{}^{m;l})$$
$$+jh^i{}_{l;m}h^{jl;m} + kh^{il;m}h^j{}_{m;l} + l(h^i{}_{l;m}h^{lm;j} + h^j{}_{l;m}h^{lm;i})$$
$$+mh_{lm}{}^{;i}h^{lm;j} + nh^i{}_l{}^{;l}h^j{}_m{}^{;m} + p(h^i{}_l{}^{;l}h_m{}^{m;j} + h^j{}_l{}^{;l}h_m{}^{m;i}) + qh_l{}^{l;i}h_m{}^{m;j}.$$

On substituting in equation (19) and collecting terms we get the identity

$$(2a+m)h^{lm;ni}h^{lm;n} + \ldots = 0.$$

If the 23 terms of this identity are independently zero, we have 23 equations for the 16 constants of t^{ij}. These are incompatible, so we conclude that for t^{ij} to exist there must be some constraint on h_{ij}, and one possibility is a gauge condition.

4.1. *The gauge condition*

In electrodynamics we impose a gauge condition $A^i{}_{;i} = 0$ which simplifies the field equation for A^i but does not otherwise affect the value of F^{ij}. In the general theory of fields Fierz (1939) gives a gauge condition $\partial A_{ik\ldots l}/\partial x_i = 0$, while both Thirring (1961) and Cavalleri and Spinelli (1974a) use a Hilbert gauge condition $h_i{}^j{}_{;j} = \frac{1}{2}h_j{}^j{}_{;i}$. If we use a more general gauge condition

$$h_i{}^j{}_{;j} = \beta h_j{}^j{}_{;i} \tag{20}$$

on the potentials, then this also affects the metric, so we must show that this is possible. We do so by means of a theorem:

Theorem. If h_{ij} and $h_{ij;k}$ are zero on an infinite outer boundary of a domain and $h_i{}^j{}_{,j} = \beta h_j{}^j{}_{;i}$ for fixed β, then the separation of g_{ij} into $b_{ij} + h_{ij}$ is unique.

Proof. Let x^{i*} be an arbitrary set of coordinates which are changed to x^i by a displacement ξ^i, where ξ_i and $\xi_{i;j}$ are zero on the outer boundary, so $x^{i*} = x^i + \xi^i$, then $g_{ij}{}^* = g_{ij} - \xi_{i;j} - \xi_{j;i}$, and if b_{ij} is constant then $h_{ij}{}^* = h_{ij} - \xi_{i;j} - \xi_{j;i}$. Hence

$$h_i{}^j{}_{;j} - \beta h_j{}^j{}_{;i} = h^*{}_i{}^j{}_{;j} - \beta h^*{}_j{}^j{}_{;i} + \xi_i{}^{;j}{}_j + \xi^j{}_{;ij} - 2\beta\xi_j{}^{;j}{}_i. \tag{21}$$

In order to make the left-hand side of equation (21) zero, we first differentiate covariantly with respect to x^i; hence

$$(2-2\beta)\Box^2\xi^i{}_{;i} = \beta h^*{}_j{}^{j;i}{}_i - h^*{}_i{}^j{}_{;j}{}^i. \tag{22}$$

The wave equation (22) with boundary condition $\xi^i{}_{;i} = 0$ on an infinite outer boundary gives $\xi^i{}_{;i}$ uniquely.

Substitute for $\xi^j{}_{;j}$ in equation (21), and again to make the left-hand side zero we get a vector wave equation

$$\Box^2\xi_i = (2\beta-1)\xi^j{}_{;ji} + \beta h^*{}_j{}^j{}_{;i} - h^*{}_i{}^j{}_{;j},$$

with $\xi_i = 0$ on the infinite outer boundary. This gives ξ_i and hence h_{ij} uniquely.

The gauge condition equation (20) may also be expressed in terms of the displacements as follows: If the gauge condition equation (20) is true for both h_{ij} and $h_{ij}{}^*$, then

$$\xi_i{}^{;j}{}_j = (2\beta-1)\xi^j{}_{;ij}. \tag{23}$$

Furthermore on differentiating covariantly by x^i

$$(2-2\beta)\xi_i{}^{;i}{}_j{}^{;j} = 0. \tag{24}$$

4.2. *The field equations with a gauge condition*

Let us impose the gauge condition on the field equations after deriving the field equations from the action, then the field equations (12) simplify to

$$2Ah^{ij;l}{}_l + [D+2\beta(B+E)]h_l{}^{l;ij} + (2C_1+\beta D_1)g^{ij}h_l{}^l{}_{;m}{}^m + \tfrac{1}{2}T^{ij} = 0.$$

For the conservation of T^{ij}, $T^{ij}{}_{;j}=0$, hence

$$[2\beta A + D + 2\beta(B+E) + 2C_1 + \beta D_1]h_l{}^l{}_{;m}{}^{mi} = 0.$$

The field equations (12) are unaltered by a displacement ξ_i if

$$2A(\xi^{i;jl}{}_l + \xi^{j;il}{}_l) + 2[D+2\beta(B+E)]\xi_l{}^{;lij} + 2(2C_i+\beta D_1)g^{ij}\xi_l{}^{;lm}{}_m = 0,$$

and, because of the limitations on ξ_i in equations (23) and (24), this reduces to

$$[2A(2\beta-1) + D + 2\beta(B+E)]2\xi_l{}^{;lij} = 0.$$

The same constraint holds for the action. There is now no constraint on μ, but as it does not appear in the field equation we may ignore it.

If we set $D+2\beta(B+E) = 2A(1-2\beta)$, $2C_1+\beta D_1 = 2A(\beta-1)$, $\mu=0$, and hence $C_1 = C$, $D_1 = D$, the field equations become

$$2A[h^{ij;l}{}_l + (1-2\beta)h_l{}^{l;ij} + (\beta-1)g^{ij}h_l{}^l{}_{;m}{}^m] + \tfrac{1}{2}T^{ij} = 0, \tag{25}$$

and the Lagrangian becomes

$$L = A(h_{lm;n}h^{lm;n} - 2h_l{}^m{}_{;m}h^l{}_n{}^{;n} + 2h_l{}^{l;m}h_{mn}{}^{;n} - h_l{}^{l;n}h^m{}_{m;n}) + B(h_{lm;n}h^{ln;m} - h_{lm}{}^{;m}h^l{}_n{}^{;n})$$
$$+(2A+B+E)(\beta^2 h_l{}^{l;n}h_m{}^m{}_{;n} - 2\beta h_l{}^l{}_{;m}h^m{}_n{}^{;n} + h_{lm}{}^{;m}h^l{}_n{}^{;n}) - \tfrac{1}{2}g_{ij}T^{ij} + \tfrac{1}{2}T.$$

This is the same as the Lagrangian in equation (16) without the gauge condition, because the second term may be made to vanish by double integration by parts and the third term vanishes because of the gauge condition. Furthermore the gauge condition may be applied wholly or partially at any stage in deriving the field equations.

4.3. *The field energy–momentum tensor*

We start with the identity (19) for the force on an element of volume, with equation (25) for T^{ij}, while t^{ij} is the most general tensor symmetrical in i and j using the gauge condition. This is

$$t^{ij} = g^{ij}(ah_{lm;n}h^{lm;n} + bh_{lm;n}h^{ln;m} + ch_l{}^{l;n}h^m{}_{m;n})$$
$$+ gh^{ij}{}_{;l}h_m{}^{m;l} + i(h^i{}_l{}^{;j} + h^j{}_l{}^{;i})h_m{}^{m;l} + jh^i{}_{l;m}h^{jl;m} + kh^{il;m}h^j{}_{m;l}$$
$$+ l(h^i{}_{l;m}h^{lm;j} + h^j{}_{l;m}h^{lm;i}) + mh_{lm}{}^{;i}h^{lm;j} + qh_l{}^{l;i}h_m{}^{m;j}.$$

On substituting in the force identity we get

$$h^{lm;ni}h_{lm;n}(2a+m)+h^{lm;ni}h_{ln;m}(2b+l)+h^{il;mn}h_{lm;n}(l+j)$$
$$+h^{il;mn}h_{mn;l}(k)+h_l^{\ l;mi}h_n^{\ n}{}_{;m}(2c+q+\beta g+\beta i)+h^{il;n}{}_n h^m{}_{m;l}(i)$$
$$+h^{il;m}h_{lm;n}{}^n(l)+h^{il;m}h^n{}_{n;lm}(g+i+\beta j+\beta k)+h^{lm;i}h_{lm;n}{}^n(m+2A)$$
$$+h^{lm;i}h^n{}_{n;lm}[i+\beta l+2A(1-2\beta)]+h_l^{\ l;i}h_m^{\ m}{}_{;n}{}^n[q+2(\beta-1)A]=0.$$

To make this expression vanish we require

$$\beta=\tfrac{1}{2},\qquad a=A,\qquad c=-A/2,\qquad m=-2A,\qquad q=A$$

and all the other coefficients zero. Finally from the measurement of the gravity field, $A=c^2/64\pi G$.

We conclude that for a linear approximation in which (i) $T^{ij}{}_{;j}=0$, (ii) the field is gauge invariant, and (iii) a field energy–momentum tensor exists, we must have a constraint on h_{ij}, and if it is a linear differential constraint it must be the Hilbert condition

$$h_i^{\ j}{}_{;j}=\tfrac{1}{2}h^j{}_{j;i}. \tag{26}$$

At the same time equations (23) and (24) reduce to $\Box^2\xi_i=0$. Then the field Lagrangian is

$$(c^2/64\pi G)(h_{lm;n}h^{lm;n}-\tfrac{1}{2}h_l^{\ l;n}h^m{}_{m;n})-\tfrac{1}{2}g^{ij}T_{ij}+\tfrac{1}{2}T, \tag{27}$$

the field equations are

$$h^{ij;l}{}_l-\tfrac{1}{2}g^{ij}h_l^{\ l}{}_{;m}{}^m+(16\pi G/c^2)T^{ij}=0 \tag{28}$$

or

$$h^{ij;l}{}_l+(16\pi G/c^2)(T^{ij}-\tfrac{1}{2}g^{ij}T)=0,$$

and the field energy–momentum tensor is

$$t^{ij}=(c^2/64\pi G)[g^{ij}(h_{lm;n}h^{lm;n}-\tfrac{1}{2}h_l^{\ l;n}h^m{}_{m;n})-2h_{lm}{}^{;i}h^{lm;j}+h_l^{\ l;i}h_m^{\ m;j}\ . \tag{29}$$

Furthermore the gauge condition (26) may be used to change some of the terms in equations (27), (28) and (29).

5. The second-order approximation and Einstein's equation

If we take a purely quadratic Lagrangian as given by equations (8) and (9) and derive the full Euler condition in equation (10) for an extremal, we find that the resulting field equations contain some non-vanishing quadratic terms as well as the linear terms. If we then proceed to work back from these field equations to derive the action by multiplying the terms other than T^{ij} in the field equations by g_{ij} and integrating by parts where necessary to find the action, we find new cubic terms in addition to the original quadratic terms.

The problem of finding a pair of tensor functions which satisfy this double condition in addition to being gauge invariant has already been solved in Einstein's equations. Let

$$S=\int g^{\mu\nu}(\Gamma_{\mu\nu}{}^{\alpha}\Gamma_{\alpha\beta}{}^{\beta}-\Gamma_{\mu\beta}{}^{\alpha}\Gamma_{\nu\alpha}{}^{\beta})(-g)^{1/2}\,\mathrm{d}^4x, \tag{30}$$

then the Euler condition for an extremal is Einstein's field equation (see Misner et al 1973, p 418)

$$R^{\mu\nu}-\tfrac{1}{2}Rg^{\mu\nu}=(8\pi G/c^2)T^{\mu\nu}.$$

Conversely Landau and Lifschitz (1975, p 269) show that $\frac{1}{2}\int g_{\mu\nu}(R^{\mu\nu}-\frac{1}{2}Rg^{\mu\nu})(-g)^{1/2}\,d^4x$ can be simplified to S of equation (30). Cavalleri and Spinelli (1975) have found more general field equations, but conclude that the additional terms are unobservable to second order.

5.1. The second-order tensor Lagrangian derived from Einstein's Lagrangian

The covariant derivative of h_{ij} is given by the equation

$$h_{ij;k}=h_{ij,k}-h_{i\alpha}\Gamma^{\alpha}_{jk}-h_{\alpha j}\Gamma^{\alpha}_{ik}.$$

Therefore

$$\begin{aligned}G_{ij,k}&=\tfrac{1}{2}(h_{ik;j}+h_{jk;i}-h_{ij;k})\\&=\tfrac{1}{2}(h_{ik,j}+h_{jk,i}-h_{ij,k})-h_{\alpha k}\Gamma^{\alpha}_{ij}\\&=\Gamma_{ij,k}-h_{\alpha k}\Gamma^{\alpha}_{ij}\end{aligned}$$

if b_{ij} is constant.

Inverting this equation to second order in h_{ij},

$$\Gamma_{ij,k}=G_{ij,k}+h^{\alpha}_{k}G_{ij,\alpha}+O(h_{ij})^3.$$

We can now substitute for the Christoffel symbols in equation (30) and simplify to second order to obtain

$$\begin{aligned}S=\int\tfrac{1}{4}(-g)^{1/2}\,d^4x\,\{&h_{lm;n}h^{lm;n}-2h_{lm;n}h^{ln;m}+2h_l{}^{l;m}h_{mn}{}^{;n}-h_l{}^{l}{}_{;n}h_m{}^{m;n}\\&+h^{\alpha\beta}[-2h_{\alpha l;m}h^{lm}{}_{;\beta}+2h_{lm;\alpha}h^{lm}{}_{;\beta}-2h_{\beta l;m}h^{lm}{}_{;\alpha}+h_l{}^{l;\alpha}(2h_{\beta m}{}^{;m}-h_m{}^{m}{}_{;\beta})\\&+h_{\alpha\beta;l}(2h^{l}{}_{m}{}^{;m}-h_m{}^{m;l})]\},\end{aligned}$$

and with the Hilbert gauge condition equation (26) this further simplifies to

$$\begin{aligned}S=\int\tfrac{1}{4}(-g)^{1/2}\,d^4x\,[&h_{lm;n}h^{lm;n}-2h_{lm;n}h^{ln;m}\\&+2h^{\alpha\beta}(h_{lm;\alpha}h^{lm}{}_{;\beta}-h_{\alpha l;m}h^{lm}{}_{;\beta}-h_{\beta l;m}h^{lm}{}_{;\alpha})].\end{aligned}$$

Similarly the second-order tensor form of the field equations is

$$\begin{aligned}&h^{ij;l}{}_{l}-\tfrac{1}{2}g^{ij}h_l{}^{l}{}_{;m}{}^{m}+\tfrac{1}{2}(h^{il}h^{j}{}_{l;m}{}^{m}+h^{jl}h^{i}{}_{l;m}{}^{m}-g^{ij}h^{lm}h_{lm;n}{}^{n})+h^{il;m}h^{j}{}_{l;m}-h^{il;m}h^{j}{}_{m;l}\\&\quad+\tfrac{1}{2}h_{lm}{}^{;i}h^{lm;j}-g^{ij}(\tfrac{3}{4}h_{lm;n}h^{lm;n}-\tfrac{1}{2}h_{lm;n}h^{ln;m})+(16\pi G/c^2)T^{ij}=0,\end{aligned}$$

or, using T to eliminate $h_l{}^{l}{}_{;m}{}^{m}$,

$$\begin{aligned}&h^{ij;l}{}_{l}+\tfrac{1}{2}(h^{il}h^{j}{}_{l;m}{}^{m}+h^{jl}h^{i}{}_{l;m}{}^{m})+h^{il;m}h^{j}{}_{l;m}-h^{il;m}h^{j}{}_{m;l}+\tfrac{1}{2}h_{lm}{}^{;i}h^{lm;j}\\&\quad+(16\pi G/c^2)(T^{ij}-\tfrac{1}{2}g^{ij}T)=0.\end{aligned}$$

6. The approximate solution for a solid sphere

The first-order gravitational field equations are

$$\Box^2 h^{ij} + (16\pi G/c^2)(T^{ij} - \tfrac{1}{2}g^{ij}T) = 0, \qquad h^j_{i;j} = \tfrac{1}{2}h^j_{j;i},$$

where $T^{00} = \rho c^2$ $(r < a)$, otherwise $T^{ij} = 0$, with boundary conditions: h^{ij} is zero at $r = \infty$, finite at $r = 0$, and continuous with continuous derivatives at $r = a$.

If we use spherical polar coordinates for the background metric and solve for a spherically symmetric metric

$$ds^2 = c^2 e^{\lambda} dt^2 - e^{\mu} dr^2 - r^2 e^{\nu}(d\theta^2 + \sin^2\theta \, d\phi^2),$$

where λ, μ and ν are small and functions of r, then to a first approximation $h_{00} = c^2\lambda$, $h_{11} = -\mu$, $h_{22} = -r^2\nu$, and $h_{33} = -r^2\nu\sin^2\theta$.

We can then use the zero-order wave equation to find the first-order approximation

$$(\lambda, \mu, \nu) = \begin{cases} (-1, 1, 1)GM(3a^2 - r^2)/c^2a^3, & r < a \\ (-1, 1, 1)2GM/c^2r, & r > a. \end{cases}$$

This solution agrees with the first-order approximation to the homogeneous solution of Einstein's equations and is in conflict with the Schwartzchild solution.

It should be noted that in two of the field equations and in the gauge condition there is a factor $(\mu - \nu)/r$ or $(\mu - \nu)/r^2$, and this factor becomes singular at the origin of r unless the constant parts of μ and ν are equal. In the complete solution $\mu = \nu$ everywhere, so the solution is homogeneous in the spatial coordinates. Also in the point particle solution this singular detail in the interior of the sphere is concealed in the monopole singularity for the point particle.

6.1. *The field energy–momentum tensor*

By substituting the sphere solution into equation (29) we find that

$$t_{ij} = \frac{GM^2r^2}{8\pi c^2a^6}\begin{bmatrix} -c^2 & & & \\ & -1 & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{bmatrix} \quad \text{and} \quad t^i_i = -\frac{GM^2r^2}{4\pi c^2a^6}, \qquad r < a,$$

while

$$t_{ij} = \frac{GM^2}{8\pi c^2r^4}\begin{bmatrix} -c^2 & & & \\ & -1 & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{bmatrix} \quad \text{and} \quad t^i_i = -\frac{GM^2}{4\pi c^2r^4}, \qquad r > a.$$

Hence the total kinetic energy of the field is $\int t_{00}\, d^3x$ integrated over all space. This is $-3GM^2/5a$, which is negative and equal to the energy lost when the mass of the sphere is condensed under its own gravitation from infinity. Furthermore the total energy is $c^2 \int t^i_i \, d^3x$ integrated over all space. This includes the stress energy and is double the kinetic energy and negative.

From the above energy–momentum tensor we also conclude that the stress field consists of a positive radial pressure and negative transverse pressures, and the resultant radial thrust on an element of volume is $t^{1j}{}_{;j}$, where

$$t^{1j}{}_{;j} = \begin{matrix} 0, & r > a \\ -6GM^2 r/4\pi c^2 a^6, & r < a. \end{matrix}$$

Thus the space outside the sphere is in equilibrium under the radial and transverse stresses, whereas the resultant force is inward inside the sphere and must be balanced by other non-gravitational forces if the sphere is in static equilibrium.

7. Conclusion

A tensor field theory of gravity may be preferable to a geometric theory because of its similarity to the nuclear and electromagnetic field theories.

It is shown here that such a theory can be developed using mechanical principles, and this leads to the same equations as were found using geometric principles.

The tensor theory has a number of similar features to vector electromagnetic field theory, and a number of important differences associated with the close link between the tensor potential and the metric tensor. Indeed this link makes a force-free geometric theory possible because b_{ij} and h_{ij} can be combined into a single g_{ij} and so the tensor potential equations may be interpreted as describing geodesics in a curved space.

It is implicit in the geometric theory that this link between b_{ij} and h_{ij} cannot be separated, and this is so in the equations of motion of a particle.

However, we find in the tensor theory that a field energy–momentum does not exist for an arbitrary metric but does exist when the metric satisfies the Hilbert gauge condition. This metric is then uniquely related to the metric in some distant outer region, assumed here to be a Minkowski metric, and is in some way a preferred metric.

In the case of the field round a static sphere the preferred metric is isotropic in the space coordinates.

Appendix 1. Some properties of tensors

(i) The commutation law for second derivatives of a tensor is similar to that for a vector, namely

$$h_{ij;kl} - h_{ij;lk} = h_{pj}R^p{}_{ikl} + h_{ip}R^p{}_{jkl},$$

where $R^p{}_{ikl}$ is the Riemann curvature tensor

$$\frac{\partial}{\partial x^k}\Gamma^p_{il} - \frac{\partial}{\partial x^l}\Gamma^p_{ik} + \Gamma^q_{il}\Gamma^p_{qk} - \Gamma^q_{ik}\Gamma^p_{ql}.$$

If $g_{ij} = b_{ij} + h_{ij}$, where b_{ij} is the Minkowski metric tensor and h_{ij} is small, then $R^p{}_{ikl}$ is comparable in size with h_{ij}, so the difference between the two covariant second derivatives of h_{ij} is comparable with $(h_{ij})^2$.

(ii) Gauss's theorem:

$$\int_R C^\lambda{}_{;\lambda}(-g)^{1/2}\,\mathrm{d}^4x = \int_R \frac{\partial}{\partial x^\lambda}[C^\lambda(-g)^{1/2}]\,\mathrm{d}^4x = \int_{\partial R_\lambda} C^\lambda\,\mathrm{d}S_\lambda$$

(iii) The rule for interchanging derivatives:

$$\int_R (h^{lm}{}_{;n}h_l{}^n{}_{;m} - h^{lm}{}_{;m}h_{ln}{}^{;n})(-g)^{1/2}\,\mathrm{d}^4x$$

$$= \int_R h_l{}^n(h^{lm}{}_{;mn} - h^{lm}{}_{;nm})(-g)^{1/2}\,\mathrm{d}^4x + \int_{\partial R_m} h^{lm}{}_{;n}h_l{}^n\,\mathrm{d}S_m - \int_{\partial R_n} h^{lm}{}_{;m}h_l{}^n\,\mathrm{d}S_n.$$

If the surface integrals vanish and the curvature tensor is comparable with h_{ij}, then the change in the integral due to interchanging the derivatives is comparable with $(h_{ij})^3$.

(iv) The Euler condition for extremalising with respect to a tensor: If $S = \int \Lambda(-g)^{1/2}\,\mathrm{d}^4x$, where $\Lambda = \Lambda(A_{\alpha\beta\ldots\delta}, A_{\alpha\beta\ldots\delta;\mu})$, then Soper (1976, p 189) shows that the Euler condition for an extremal has a covariant form

$$\frac{\partial\Lambda}{\partial A_{\alpha\beta\ldots\delta}} = \left(\frac{\partial\Lambda}{\partial A_{\alpha\beta\ldots\delta;\mu}}\right)_{;\mu}$$

However, if the metric tensor is also a function of the tensor potential, $g_{\mu\nu} = b_{\mu\nu} + h_{\mu\nu}$, $g = \det(g_{\mu\nu})$ and $\Lambda = \Lambda(g^{\mu\nu}, h_{\mu\nu}, h_{\lambda\mu;\nu})$, then

$$\frac{\partial g_{\mu\nu}}{\partial h_{\mu\nu}} = 1, \qquad \frac{\partial g^{\alpha\beta}}{\partial h_{\mu\nu}} = -g^{\alpha\mu}g^{\beta\nu}, \qquad \frac{1}{g}\frac{\partial g}{\partial h_{\mu\nu}} = g^{\mu\nu},$$

and the covariant form of Euler's equation reduces to

$$\left(\frac{\partial\Lambda}{\partial h_{\alpha\beta;\mu}}\right)_{;\mu} = \frac{\partial\Lambda}{\partial h_{\alpha\beta}} - g^{\alpha\mu}g^{\beta\nu}\frac{\partial\Lambda}{\partial g^{\mu\nu}} + \frac{1}{2}g^{\alpha\beta}\Lambda + \frac{1}{2}\left(h_l{}^\alpha\frac{\partial\Lambda}{\partial h_{l\beta;k}} + h_l{}^\beta\frac{\partial\Lambda}{\partial h_{l\alpha;k}}\right.$$

$$\left. - h_l{}^k\frac{\partial\Lambda}{\partial h_{l\alpha;\beta}} - h_l{}^k\frac{\partial\Lambda}{\partial h_{l\beta;\alpha}} + h_l{}^\alpha\frac{\partial\Lambda}{\partial h_{kl;\beta}} + h_l{}^\beta\frac{\partial\Lambda}{\partial h_{kl;\alpha}}\right)_{;k}.$$

Appendix 2. Local and distant observations

An example of a local phenomenon is the motion of a particle as described by equations (2), (3) or (4) or the physical measurement of a time or distance using the metric equation $\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$. In these equations the gravitational potential is an unobservable part of the metric tensor.

A third local phenomenon is the measurement of the velocity of light. If standard clocks measure $\mathrm{d}s/c$ when $\mathrm{d}s^2$ is positive, while standard rods measure $\mathrm{i}\,\mathrm{d}s$ when $\mathrm{d}s^2$ is negative, then in terms of local time T and local lengths X, Y, Z the metric equation can be transformed to

$$\mathrm{d}s^2 = c^2\,\mathrm{d}T^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 - \mathrm{d}Z^2,$$

and in particular for the null geodesic, $\mathrm{d}s^2 = 0$, so $c^2 = (\mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2)/\mathrm{d}T^2$. Hence the fundamental velocity c is the same for all observers in any direction with any acceleration anywhere in a gravitational field provided it is measured with local standardised rods and clocks.

These are examples of the general postulate that local observers are unable to detect any gravitational effects or preferred metrics or preferred directions in space.

On the other hand there are a number of distant observations which enable us to measure variations in the gravitational field:

Let the metric for the neighbourhood of the sun in polar background coordinates (t, r, θ) be

$$ds^2 = g_{00}\,dt^2 + g_{11}\,dr^2 + g_{22}\,d\theta^2$$
$$= c^2(1 - 2a/r + \ldots)\,dt^2 - (1 + 2b/r + \ldots)\,dr^2 - (r^2 + 2\,dr + \ldots)\,d\theta^2$$

where a, b and d are constants.

(i) If we measure the accelerations of falling bodies in different places near the sun, we cannot distinguish between a constant acceleration and a constant force field but we can observe a non-uniform residual which enables us to determine the constant a.

(ii) We may compare distant clocks. If two observers P_1 and P_2 at r_1 and r_2 time the same physical process with their own local clocks, and during the process they remain relatively stationary while only the parameter t alters, then the time recorded by P_1 is ds_1/c and the time recorded by P_2 is ds_2/c, where

$$\frac{ds_1/c}{(1 - 2a/r_1)^{1/2}} = \frac{ds_2/c}{(1 - 2a/r_2)^{1/2}} = dt.$$

This is a second method for calculating g_{00} or a.

(iii) The bending of light is a distant process. The deflection of a light ray passing within a distance R of the centre is $2(a + b)/R$.

(iv) The number of wavelengths of light in a light path is an invariant for all observers. For a light path from radius r_1 passing within a radius r_0 of the centre and continuing to radius r_2 beyond the nearpoint, the time parameter t along the path is

$$\frac{1}{c}\Bigg\{(r_1^2 - r_0^2)^{1/2} + (r_2^2 - r_0^2)^{1/2} + (a + b)\ln\left(\frac{[r_1 + (r_1^2 - r_0^2)^{1/2}][r_2 + (r_2^2 - r_0^2)^{1/2}]}{r_0^2}\right)$$
$$+ (a + d)\left[\left(\frac{r_1 - r_0}{r_1 + r_0}\right)^{1/2} + \left(\frac{r_2 - r_0}{r_2 + r_0}\right)^{1/2}\right]\Bigg\}.$$

The constant d in this equation is related to the radial parameter r in such a way as to make this independent of the choice of radial parameter. Hence this measurement only measures d if r and t are appropriately defined.

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